Picard's theorem

We consider complex analytic functions $f(z)$ in the complex plane $\mathbb{C}$, where $z = x+iy$, $i^2=-1$.

We recall the Liouville's theorem which states that

If a entire (everywhere differentiable/analytic/holomorphic) function which is bounded, then $f(z) \equiv$ constant on $\mathbb{C}$.

In 1879, E. Picard (1856–1941) extended Liouville's theorem:

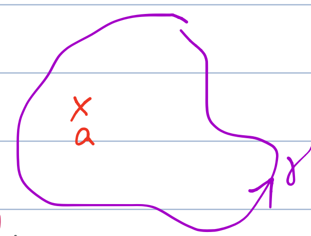
Little Picard's theorem: An entire function f omitting two distinct complex numbers must be constant.

Eg. $f(z) = e^z$ omits $z=0$.

We present a proof due to B. Q. Li published in 2016 Journal of Mathematical Analysis and Applications, 442, pp. 446–450.

We recall Cauchy's integral formula for an analytic function $f(z)$ defined on an simply connected domain D in $\mathbb{C}$. If $a \in D$ and γ is a simple closed contour in D such that $a \notin \gamma$. If $n(\gamma;a) \neq 0$, then

$$n(\gamma;a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{s-a} \, ds$$



We usually have $n(\gamma;a) = 1$ and $\gamma(\theta) = a + Re^{i\theta}$, $\theta \in [0, 2\pi]$. This statement is equivalent to Cauchy's theorem that

$$\int_\gamma f(s) \, ds = 0$$

where $f(z)$ is analytic in the interior of the contour γ.

In Cauchy's integral formula, we choose $\gamma(t) = a + re^{it}$.

Then we have

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) \, d\theta. \qquad (1)$$

Theorem 1 Let $f(z)$ be analytic in $|z|\le R$ and $A(r,f)=\max_{|z|\le r} \mathrm{Re}(f(z))$.

Then for $0<r<R$ and $n\ge 1$,

$$\frac{1}{2\pi}\int_0^{2\pi}|f^{(n)}(re^{i\theta})|\,d\theta \le \frac{2n!}{(R-r)^n}\left[A(R,f)-\mathrm{Re}(f(0))\right]. \quad (2)$$

Proof: For any $|w|\le r$, Cauchy's theorem implies

$$0=\int_{|z-w|=\rho} f(z)(z-w)^{n-1}dz = i\rho^n\int_0^{2\pi} f(w+\rho e^{i\theta})\,e^{in\theta}\,d\theta,$$

where $\rho\le R-r$. Similarly

$$\int_0^{2\pi}\overline{f(w+\rho e^{i\theta})e^{in\theta}}\,d\theta=0.$$

Hence

$$\begin{aligned}
f^{(n)}(w) &= \frac{n!}{2\pi i}\int_{|z-w|=\rho}\frac{f(s)}{(s-w)^{n+1}}ds\\
&= \frac{n!}{2\rho^n\pi}\int_0^{2\pi} f(w+\rho e^{i\theta})e^{-in\theta}d\theta + 0\\
&= \frac{n!}{2\rho^n\pi}\int_0^{2\pi} f(w+\rho e^{i\theta})e^{-in\theta}d\theta + \frac{n!}{2\rho^n\pi}\int_0^{2\pi}\overline{f(w+\rho e^{i\theta})e^{-in\theta}}\,d\theta\\
&= \frac{n!}{\rho^n\pi}\int_0^{2\pi}\mathrm{Re}\left[f(w+\rho e^{i\theta})e^{-in\theta}\right]d\theta.
\end{aligned}$$

We replace f by $f(s)-A(R,f)$ in the above inequality. Let $\rho=R-r$.

Notice that $A(R,f) - \mathrm{Re}(f(w+\rho e^{i\theta})) \geq 0$. Hence

$$f^{(n)}(w) = [f(w) - A(R,f)]^{(n)} \leq \frac{n!}{(R-r)^n \pi} \int_0^{2\pi} A(R,f) - \mathrm{Re}(f(w+\rho e^{i\theta}))\, d\theta$$

$$= \frac{2n!}{(R-r)^n}\left(A(R,f) - \mathrm{Re}(f(w))\right)$$

constant

We note that

$$f^{(n)}(w) = \frac{n!}{\rho^n \pi} \int_0^{2\pi} \mathrm{Re}\left[f(w+\rho e^{i\theta}) e^{-in\theta}\right] d\theta.$$

yields

$$|f^{(n)}(w)| \leq \frac{n!}{\rho^n \pi} \int_0^{2\pi} \left|\mathrm{Re}\left[f(w+\rho e^{i\theta})\right]\right| d\theta. \quad \text{— (3)}$$

Hence if $f(z)$ is zero-free in $|z| \leq R$, then $\mathrm{Log}\, f(w)$ is well-defined and so

$$|\mathrm{Log}\, f(w)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log|f(\rho e^{i\theta})|\, d\theta \quad \text{———— (4)}$$

Exercise: Deduce the following inequality when f is zero-free in $|z| \leq R$, $0 < r < R$:

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta \leq \frac{2}{R-r}\Big(\log M(R,f) - \log|f(0)|\Big) \quad - (5)$$

where $M(r,f) = \max_{|z|=r} |f(z)|$ is the maximum modulus of the holomorphic function $f(z)$.

Exercise Prove that the maximum modulus is a non-decreasing function of r.

Exercise A real-valued function is a convex function on $[a,b]$ if

$$f(tx+(1-t)y) \leq t f(x) + (1-t) f(y)$$

for all x, y in $[a,b]$.

Prove that $\log x$ is a convex function of $x (>0)$.

Then show that

$$\int_0^{2\pi} \log g(x)\, dx \leq \log \int_0^{2\pi} g(x)\, dx.$$

Exercise Let $\log^+ x = \max\{0, \log x\}$. Show

$$\left.\begin{array}{l} \text{(i)}\ \log^+(xy) \leq \log^+ x + \log^+(y) \\ \text{(ii)}\ \log^+(x+y) \leq \log^+ x + \log^+ y + \log 2 \end{array}\right\} \ x, y > 0.$$

Exercise Use the hint

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \log\left(\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|+1\right) d\theta$$

$$\leq \log\left[\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|+1\, d\theta\right] \leq \log^+\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta+\log 2$$

(why?)

to show that

Exercise if $f(z)$ has no zeros in $|z|\leq R$, $0<r<R$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta \leq \log^+\frac{1}{R-r}+\log^+\log^+ M(R,f)+\log^+\log^+\frac{1}{|f(0)|}$$

$$+\,3\log 2. \qquad (6)$$

We defined a new real valued function by

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta.$$

Let $|w|=\rho<R$. Apply the Cauchy integral formula to the function

$$\log f(z)\cdot\frac{R^2-w^2}{R^2-z\bar{w}}$$

to show

Exercise

$$\log f(w)=\frac{1}{2\pi i}\int_{|z|=R}\frac{\log f(z)}{z-w}\cdot\frac{R^2-w^2}{z\bar{z}-z\bar{w}}\,dz=\frac{1}{2\pi}\int_0^{2\pi}\log f(Re^{i\theta})\frac{R^2-|w|^2}{|Re^{i\theta}-w|^2}\,d\theta$$

The show

Exercise $\log M(R,f) \leq \frac{R+\rho}{R-\rho} m(R,f)$

Let $\rho = \frac{R+r}{2}$ $(< R)$. Deduce from the inequality ( ) that

Exercise

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta$$

$$\leq 2\log^+ \frac{1}{R-r} + \log^+ R + \log^+ m(R,f) + \log^+\log^+ \frac{1}{|f(0)|} + 5\log 2, \qquad (7)$$

where again $M(r,f) = \max_{|z|\leq r} |f(z)|$.

Exercise Show that the map

$$\frac{r(z-a)}{z\bar{z} - \bar{a}z}, \qquad |a| < r$$

maps $|z| < 1$ onto (and one-one) $|z| < r$ with $|z|=1$ to $|z| = r$.

Exercise Suppose $f(z)$ is holomorphic in $|z| < r$, $f(0) \neq 0$. Then for $r > 0$,

$$\log |f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \qquad (8)$$

(Hint: Consider the function $f(z) \times \prod_{j=1}^{k} \frac{r^2 - \bar{a}_j z}{r(r-a_j)}$ where $a_1, \ldots, a_k$ are zeros in $|z| < r$)

Exercise Let $\phi(r)>0$ be a function of $r \geq r_0$ such that ϕ is bounded on $[r_0, r]$ for large r. Let $A>1$. Prove that there is a sequence $\{r_n\} \uparrow \infty$ such that

$$\phi\left(r_n + \frac{1}{\phi(r_n)}\right) < A\,\phi(r_n) \qquad (9)$$

(Hint: Suppose on the contrary that $\exists\, r_1 > r_0$ such that $\phi(r + 1/\phi(r)) \geq A\,\phi(r)$ for all $r \geq r_1$. Set $r_{n+1} = r_n + 1/\phi(r_n)$ for $n \geq 1$. Then $\phi(r_n) = \phi(r_{n-1} + 1/\phi(r_{n-1})) \geq A\,\phi(r_{n-1}) \geq A^2 \phi(r_{n-2}) \geq \cdots \geq A^{n-1}\phi(r_1)$. Then estimate the sum $\sum_{n=2}^{\infty}(r_{n+1} - r_n)$ to obtain a contradiction.)

Theorem (Little Picard theorem) An entire function f omitting two distinct complex numbers must be constant.

Proof Without loss of generality, we may assume $f(z) \neq 0, 1$. For if $f \neq a, b$, then $F(z) := \dfrac{f(z)-a}{b-a} \neq 0, 1$. Consider

$$F(z) = \frac{(f-1)'}{f-1} - \frac{f'}{f} = \frac{f'}{f(f-1)} = \frac{d}{dz}\log\left(1 - \frac{1}{f}\right)$$

Apply the identity $\log x = \log^+ x - \log^+ \frac{1}{x}$ and (8) to show

$$m\left(r, \frac{1}{F}\right) = m(r, F) - \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\, d\theta \leq m(r, F) + O(1).$$

Then apply (6) and (7) to show

Exercise

$$m(r,f) \le m(r, \tfrac{1}{f}) + m(r, \tfrac{f'}{f-1}) \le m(r,F) + m(r, \tfrac{f'}{f-1}, r)$$

$$\le 6 \log^+ \frac{1}{R-r} + 3\log^+ R + 3\log^+ m(R,f) + O(1)$$

Exercise Now set $\phi(r) = \log m(r,f)$ in (9) to obtain a sequence r_n such that $m(R_n, f) = 2\log m(r_n, f)$, where $R_n = r_n + 1/\phi(r_n)$.

Exercise Apply (3) to $\log f$ to deduce that

$$\left|\frac{f'(0)}{f(0)}\right| \le \frac{1}{r_n \pi} \int_0^{2\pi} \left|\log|f(r_n e^{i\theta})|\right| d\theta \le \frac{2}{r_n}\left(m(r_n,f) + m(r_n, \tfrac{1}{f})\right)$$

$$\le \frac{2}{r_n}\left(2m(r_n,f) + O(1)\right) = \frac{O(\log r_n)}{r_n} \longrightarrow 0, \; n \longrightarrow \infty$$

and $f'(0) = 0$. Modify the above argument to $f(s+z)$ to show that $f'(s) = 0$ for all z. Hence show that $f \equiv$ constant.

Theorem 1. *Let f be a holomorphic function in $|z| \leq R$. Then for $0 < r < R$ and $n \geq 1$,*

$$\frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|d\theta \leq \frac{2n!}{(R-r)^n}\big(A(R,f) - Re\{f(0)\}\big), \qquad (1)$$

where $A(r,f) = \max_{|z|\leq r}\{Re\{f(z)\}\}$.

⋮

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})d\theta. \qquad (5)$$

Proof of Theorem 1. For any w with $|w| \leq r$, we have by Cauchy's theorem that

$$0 = \int_{|z-w|=\rho} f(z)(z-w)^{n-1}dz = i\rho^n \int_0^{2\pi} f(w+\rho e^{i\theta})e^{in\theta}d\theta,$$

where $\rho \leq R-r$. Thus, $\int_0^{2\pi} \overline{f(w+\rho e^{i\theta})e^{in\theta}}d\theta = 0$. From this and Cauchy's formula (after taking derivatives),

$$\begin{aligned}
f^{(n)}(w) &= \frac{n!}{2\pi i}\int_{|z-w|=\rho} \frac{f(z)}{(z-w)^{n+1}}dz \\
&= \frac{n!}{2\rho^n\pi}\int_0^{2\pi} f(w+\rho e^{i\theta})e^{-in\theta}d\theta + 0 \\
&= \frac{n!}{2\rho^n\pi}\int_0^{2\pi} f(w+\rho e^{i\theta})e^{-in\theta}d\theta + \frac{n!}{2\rho^n\pi}\int_0^{2\pi} \overline{f(w+\rho e^{i\theta})e^{in\theta}}d\theta \\
&= \frac{n!}{\rho^n\pi}\int_0^{2\pi} \mathrm{Re}\{f(w+\rho e^{i\theta})\}e^{-in\theta}d\theta \qquad (6)
\end{aligned}$$

Applying this result to the function $f - A(R,f)$ and noting that $A(R,f) - \mathrm{Re}\{f(w+\rho e^{i\theta})\} \geq 0$, we deduce that with $\rho = R-r$,

$$\begin{aligned}
|f^{(n)}(w)| &\leq \frac{n!}{(R-r)^n\pi}\int_0^{2\pi} \big(A(R,f) - \mathrm{Re}\{f(w+\rho e^{i\theta})\}\big)d\theta \\
&= \frac{2n!}{(R-r)^n}\big(A(R,f) - \mathrm{Re}\{f(w)\}\big)
\end{aligned}$$

by virtue of (5) (after taking the real parts), and thus $\frac{1}{2\pi}\int_0^{2\pi} |f^{(n)}(re^{i\theta})|d\theta \leq \frac{2n!}{(R-r)^n}\big(A(R,f) - \mathrm{Re}\{f(0)\}\big)$ by (5) again. □